Answer Quiz (2) First year Electrical Department (Communication Branch)

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(1) Prove that $\Gamma(n+1) = n\Gamma(n)$

Answer $\because \Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt$ $\qquad \therefore \Gamma(n+1) = \int_0^\infty e^{-t}\, t^{n}\, dt$

Integrate by parts put $u = t^n \qquad du = nt^{n-1}dt$

$$dv = e^{-t}dt \qquad v = -e^{-t}$$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-t}\, t^n\, dt = -\left[t^n e^{-t}\right]_0^\infty + \int_0^\infty e^{-t}\, nt^{n-1}\, dt = n\int_0^\infty e^{-t}\, t^{n-1}\, dt = n\,\Gamma(n)$$

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(2) Evaluate $\int_0^\infty \frac{x^2}{1+x^4}dx$

Answer: put $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta} \to dx = \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta$

when $x=0$ we find $\theta=0$ and when $x=\infty$ we find $\theta=\frac{\pi}{2}$

$$\int_0^\infty \frac{x^2}{1+x^4}dx = \int_0^{\pi/2} \frac{\tan^2\theta}{(1+\tan^2\theta)}\frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta = \frac{1}{2}\int_0^{\pi/2}\sqrt{\tan\theta}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta d\theta = \frac{1}{4}\mathrm{B}(\tfrac{3}{4},\tfrac{1}{4}) = \frac{1}{4}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)} = \frac{1}{4}.\frac{\pi}{\sin(\pi/4)} = \frac{\sqrt{2}\pi}{4} = \frac{\pi}{2\sqrt{2}}$$

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(1) **Evaluate** $\int_0^{\pi/2}\cos^3\theta\sin^7 d\theta$

Answer: $\int_0^{\pi/2}\cos^3\theta\sin^7 d\theta = \frac{1}{2}\mathrm{B}(4/2, 8/2) = \frac{1}{2}\mathrm{B}(2,4) = \frac{1}{2}\frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{3!}{(2)(5!)}$

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(4)Since $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

$$\Gamma(-\tfrac{7}{2}) = \frac{1}{(-\frac{7}{2})}\Gamma(-\tfrac{5}{2}) = \frac{1}{(-\frac{7}{2})}\frac{1}{(-\frac{5}{2})}\Gamma(-\tfrac{3}{2}) = \frac{1}{(-\frac{7}{2})}\frac{1}{(-\frac{5}{2})(-\frac{3}{2})}\Gamma(-\tfrac{1}{2}) = \frac{16}{105}\sqrt{\pi}$$